

Perimeter as relaxed Minkowski content in metric measure spaces

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Abstract

In this note we prove that on general metric measure spaces the perimeter is equal to the relaxation of the Minkowski content w.r.t. convergence in measure.

1 Introduction

In a metric measure space (X, d, \mathbf{m}) , the upper and lower Minkowski contents are respectively defined by

$$\mathcal{M}_-(A) := \liminf_{r \downarrow 0} \frac{\mathbf{m}(A^r) - \mathbf{m}(A)}{r}, \quad \mathcal{M}_+(A) := \limsup_{r \downarrow 0} \frac{\mathbf{m}(A^r) - \mathbf{m}(A)}{r}$$

for Borel sets A with finite \mathbf{m} -measure. The Minkowski contents, even in their non-infinitesimal versions, appear in many areas, as in the theory of concentration of measures and isoperimetric inequalities (see for instance [Led], [CM15a], [CM15b] and the references therein) and the theory of random closed sets [AmCaVi].

For sufficiently nice metric measure structures, the relations between Minkowski content and perimeter are well-known, see for instance §2.13 in [AFP] and §14.2 in [BuZa]. Aim of this note is the investigation of more precise relations between the Minkowski content and the perimeter, as defined in the theory of BV functions in metric measure spaces. In particular we prove in Lemma 2.1 and Theorem 3.6 that the lower semicontinuous envelope w.r.t. $L^1(X, \mathbf{m})$ convergence of the Minkowski contents $\mathcal{M}_\pm(A)$ is equal to the perimeter $\text{Per}(A)$. As a byproduct, we can prove that in metric measure spaces with finite \mathbf{m} -measure the Cheeger constant

$$\gamma := \inf \left\{ \frac{\mathcal{M}_+(A)}{\mathbf{m}(A)} : 0 < \mathbf{m}(A) \leq \frac{\mathbf{m}(X)}{2} \right\}$$

can be equivalently defined replacing $\mathcal{M}_+(A)$ with $\mathcal{M}_-(A)$, or with $\text{Per}(A)$.

Another consequence of our result is that whenever one wants to establish, on a given space, an isoperimetric inequality of the form

$$\mathbf{m}(A) \leq f(\text{Per}(A))$$

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for some continuous non-decreasing function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, it is sufficient to prove the easier

$$\mathbf{m}(A) \leq f(\mathcal{M}_+(A)).$$

We remark that not only it is easily seen that $\mathcal{M}_+(A) \geq \text{Per}(A)$ (see also the proof of Theorem 3.6), but also that the Minkowski content is a quantity that by nature is sometimes handled better than the perimeter in estimates involving the geometry of the space. An example in this direction is the recent paper [CM15a] by Cavalletti-Mondino, which motivated our study.

Another goal of the paper is a closer investigation of the coarea formula and of the “generic” properties of superlevel sets of Lipschitz functions. In Euclidean and other nice spaces, the combination of the Fleming-Rishel formula (involving the perimeter of superlevel sets) with the coarea formula for Lipschitz maps (involving the Hausdorff measure of level sets) provides many useful informations, even on the level sets, as illustrated in Remark 4.1. Under a suitable regularity assumption (4.2) on the metric measure structure, fulfilled in all spaces $RCD(K, \infty)$ of [AGS11b], we provide in Proposition 4.2 a metric counterpart of this, involving the Minkowski contents. Finally, we are able to make a more detailed analysis for level sets of distance functions and we conclude the paper pointing out a few open questions.

2 Basic setting and preliminaries

Throughout this paper (X, d) is a metric space and \mathbf{m} is a nonnegative and σ -additive measure on its Borel σ -algebra; we always assume that \mathbf{m} is finite on bounded Borel sets. In particular, \mathbf{m} is σ -finite.

In the metric space (X, d) we denote by $B_r(x)$ the open ball with center x and radius r . We denote by $\text{Lip}(f)$ the Lipschitz constant of a Lipschitz function $f : X \rightarrow \mathbb{R}$ and we will often use the distance function

$$d_A(x) := \inf_{y \in A} d(x, y)$$

from a nonempty set A , whose Lipschitz constant is less than 1. The slope $\text{lip}(f)$ (also called local Lipschitz constant) of $f : X \rightarrow \mathbb{R}$ is defined by

$$\text{lip}(f)(x) := \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)},$$

with the convention $\text{lip}(f)(x) = 0$ if x is an isolated point.

We denote by $\chi_A : X \rightarrow \{0, 1\}$ the characteristic function of a set A and we say that $A_h \rightarrow A$ in \mathbf{m} -measure if $\int_X |\chi_{A_h} - \chi_A| d\mathbf{m} \rightarrow 0$ (equivalently, $\mathbf{m}(A_h \Delta A) \rightarrow 0$). For any nonempty set $A \subset X$ and any $r > 0$ we define the open r -enlargement A^r of A by

$$A^r := \{x \in X : d_A(x) < r\}.$$

Notice that $A^r = (\overline{A})^r$, and that the triangle inequality gives the semigroup inclusion

$$(A^s)^t \subset A^{s+t} \quad s, t > 0. \tag{2.1}$$

For any Borel set $A \subset X$ with $\mathbf{m}(A) < \infty$ we define the upper and lower Minkowski contents by

$$\mathcal{M}_-(A) := \liminf_{r \downarrow 0} \frac{\mathbf{m}(A^r) - \mathbf{m}(A)}{r}, \quad \mathcal{M}_+(A) := \limsup_{r \downarrow 0} \frac{\mathbf{m}(A^r) - \mathbf{m}(A)}{r}.$$

Obviously one has $0 \leq \mathcal{M}_-(A) \leq \mathcal{M}_+(A) \leq \infty$. In addition, $\mathcal{M}_-(A) = \infty$ if $\mathbf{m}(\overline{A} \setminus A) > 0$, hence a natural domain for the Minkowski content is the class of essentially closed sets. Notice also that $\mathcal{M}_-(A) < \infty$ implies $\mathbf{m}(A^r \setminus A) \rightarrow 0$ as $r \downarrow 0$.

We define the relaxed Minkowski contents $\mathcal{M}_-^*(A)$, $\mathcal{M}_+^*(A)$ of A by

$$\begin{aligned}\mathcal{M}_-^*(A) &:= \inf \left\{ \liminf_{h \rightarrow \infty} \mathcal{M}_-(A_h) : A_h \rightarrow A \text{ in } \mathbf{m}\text{-measure} \right\}, \\ \mathcal{M}_+^*(A) &:= \inf \left\{ \liminf_{h \rightarrow \infty} \mathcal{M}_+(A_h) : A_h \rightarrow A \text{ in } \mathbf{m}\text{-measure} \right\}.\end{aligned}$$

It is obvious that $\mathcal{M}_-^*(A) \leq \mathcal{M}_+^*(A)$. The following elementary lemma shows that equality holds, so from now on we will also use the notation $\mathcal{M}^*(A)$ for their common value.

Lemma 2.1 *For any Borel set $A \subset X$ one has*

$$\mathcal{M}_-(A) \geq \mathcal{M}_+^*(A). \quad (2.2)$$

In particular $\mathcal{M}_-^(A) = \mathcal{M}_+^*(A)$.*

Proof. We can assume with no loss of generality $\mathcal{M}_-(A) < \infty$, hence $\mathbf{m}(A^s \setminus A) \rightarrow 0$ as $s \downarrow 0$. We start from the semigroup inclusion (2.1) to get

$$\frac{\mathbf{m}(A^{s+t}) - \mathbf{m}(A^s)}{t} \geq \frac{\mathbf{m}((A^s)^t) - \mathbf{m}(A^s)}{t}.$$

Now, at any differentiability point s of the nondecreasing and left continuous map $f(r) := \mathbf{m}(A^r)$ we get $f'(s) \geq \mathcal{M}_+(A^s)$ and an integration w.r.t. s gives

$$\mathbf{m}(A^r) - \mathbf{m}(A) = f(r) - f(0_+) \geq \int_0^r f'(s) ds \geq \int_0^r \mathcal{M}_+(A^s) ds \quad \forall r > 0.$$

Dividing both sides by r and letting $r \downarrow 0$ an application of the mean value theorem gives the first claim.

In order to prove the equality of relaxed Minkowski contents, let $A_h \rightarrow A$ in \mathbf{m} -measure and apply (2.2) to get

$$\mathcal{M}_+^*(A) \leq \liminf_{h \rightarrow \infty} \mathcal{M}_+^*(A_h) \leq \liminf_{h \rightarrow \infty} \mathcal{M}_-(A_h).$$

The arbitrariness of A_h yields the inequality $\mathcal{M}_+^*(A) \leq \mathcal{M}_-^*(A)$. □

3 Equality of relaxed contents and perimeter

In this section we shall use the semigroup

$$T_t f(x) := \sup_{B_t(x)} f, \quad T_0 f = f, \quad (3.1)$$

which extends the operator $A \mapsto A^t$ from characteristic functions to functions, i.e. $T_t \chi_A = \chi_{A^t}$ for all $t > 0$. Actually the term “semigroup” is not totally correct here, since only the inequality $T_{s+t} f \geq T_s(T_t f)$ holds for the operator T_t in (3.1); the inequality is an equality

if (X, \mathbf{d}) is a length space (see also [AmDi] for the simple proof and an example of strict inequality). We will only need the inequality in the sequel.

Notice that $T_t f \geq f$ and that $T_t f$ is lower semicontinuous for all $t > 0$. The definition of $\text{lip}(f)$ gives also immediately

$$\limsup_{t \downarrow 0} \frac{T_t f - f}{t} \leq \text{lip}(f). \quad (3.2)$$

Remark 3.1 The semigroup T_t already had a role (up to a change of sign, resulting in the replacement of \sup with \inf) in the proof given in [AmDi] of the equality between relaxed upper gradients and “measure” upper gradients in the context of the theory of BV functions and sets of finite perimeter. The “inf” semigroup can also be formally viewed as the limit as $p \rightarrow \infty$ of the semigroups

$$T_t^p f(x) := \inf_{y \in X} f(y) + \frac{1}{pt^{p-1}} \mathbf{d}^p(y, x).$$

associated to the Hamilton-Jacobi equation $\partial_t g + |\nabla g|^q/q = 0$, see [AGS11a, GRS].

It is also worthwhile to mention that the semigroup T_t has already been used in connection with the (anisotropic) Minkowski content in [CLL], but only in an Euclidean setting: their case can be regarded as our construction in the metric space $(\mathbb{R}^d, \mathbf{d}_C, \mathcal{L}^d)$, where \mathbf{d}_C is the gauge function associated to a convex set C . ■

Lemma 3.2 *For all $f : X \rightarrow [0, \infty)$ Lipschitz with $\mathbf{m}(\{f > 0\}^h) < \infty$ for some $h > 0$, one has*

$$\int_0^\infty \mathcal{M}_-(\{f \geq t\}) dt \leq \liminf_{t \downarrow 0} \int_X \frac{T_t f - f}{t} d\mathbf{m} \leq \int_X \text{lip}(f) d\mathbf{m}. \quad (3.3)$$

Proof. We start from the elementary “coarea” pointwise identities $\int_0^\infty \chi_{\{f \geq t\}} dt = f$ and

$$\int_0^\infty T_h \chi_{\{f \geq t\}} dt = \int_0^\infty \chi_{\{f \geq t\}^h} dt = T_h f$$

for $f : X \rightarrow [0, \infty]$. If f is Borel with $\int_X f d\mathbf{m} < \infty$, subtracting the first identity from the second one and integrating w.r.t. \mathbf{m} gives

$$\int_0^\infty \frac{\mathbf{m}(\{f \geq t\}^h) - \mathbf{m}(\{f \geq t\})}{h} dt = \int \frac{T_h f - f}{h} d\mathbf{m}.$$

Now, notice that one has the bound

$$\frac{T_h f - f}{h} \leq \text{Lip}(f) \chi_{\{f > 0\}^h}.$$

Hence, assuming in addition that f is Lipschitz and that for some $h > 0$ we have $\mathbf{m}(\{f > 0\}^h) < \infty$, Fatou’s lemma, dominated convergence and (3.2) give (3.3). □

The following definition of perimeter, first proposed in [Mi] and further investigated in [AmDi] (dropping the local compactness assumption on the metric structure) is by now well studied, see also the following remarks.

Definition 3.3 (Perimeter) *Let $A \subset X$ be a Borel set with $\mathbf{m}(A) < \infty$. We define*

$$\text{Per}(A) := \inf \left\{ \liminf_{h \rightarrow \infty} \int_X \text{lip}(f_h) d\mathbf{m} : f_h \in \text{Lip}(X), \lim_{h \rightarrow \infty} \int_X |f_h - \chi_A| d\mathbf{m} = 0 \right\}. \quad (3.4)$$

If $\mathbf{m}(X)$ is finite we can also consider the perimeter of $X \setminus A$; it is then easy to check that $\text{Per}(A) = \text{Per}(X \setminus A)$. In the following remarks we show that smaller classes than $\text{Lip}(X)$ can be considered in the definition of $\text{Per}(A)$, comparing also with the definitions in [Mi] and [AmDi].

Remark 3.4 A simple truncation argument shows that we need only to consider sequences f_h with $0 \leq f_h \leq 1$ in (3.4). Also, if $\bar{x} \in X$ and $\phi_R : X \rightarrow [0, 1]$ are 1-Lipschitz functions with $\phi_R \equiv 1$ on $B_R(\bar{x})$ and $\phi_R \equiv 0$ on $X \setminus B_{2R}(\bar{x})$, from the inequality

$$\int_X \text{lip}(f_h \phi_R) d\mathbf{m} \leq \int_X \text{lip}(f_h) d\mathbf{m} + \int_{X \setminus B_R(\bar{x})} f_h d\mathbf{m}$$

and a diagonal argument (i.e. choosing $R = R(h)$) we obtain that only sequences of Lipschitz functions $f_h : X \rightarrow [0, 1]$ with $\{f_h > 0\}$ bounded need to be considered in the definition of $\text{Per}(A)$. \blacksquare

Remark 3.5 Definition 3.3 appeared first in [Mi] in complete and doubling metric measure spaces, requiring approximation by locally Lipschitz functions and L^1_{loc} convergence. Then, at the general level of complete and separable metric spaces (for instance the most appropriate setting for the theory of $CD(K, \infty)$ spaces), the definition has been revisited in [AmDi], requiring L^1 convergence of f_h to χ_A , as in (3.4), but requiring f_h to be locally Lipschitz (i.e. for any $x \in X$ there exists $r > 0$ such that $f|_{B_r(x)}$ is Lipschitz). As in [Mi], the motivation for this definition is that, when localized to open sets $U \subset X$ via

$$\mu(U) := \inf \left\{ \liminf_{h \rightarrow \infty} \int_U \text{lip}(f_h) d\mathbf{m} : f_h \in \text{Lip}_{\text{loc}}(U), \lim_{h \rightarrow \infty} \int_U |f_h - \chi_A| d\mathbf{m} = 0 \right\}$$

is continuous w.r.t. monotone nondecreasing sequences and thus it provides the restriction to open sets of a finite Borel measure, namely the perimeter measure.

In proper metric spaces (a class that includes complete and doubling metric spaces) it is not hard to prove that all these variants of Definition 3.3 lead to the same definition, for sets with finite \mathbf{m} -measure. The equivalence persists also for complete and separable spaces, but the proof is not elementary, see Section 4 and, in particular, Theorem 4.5.3 in [Di]. However, the equivalence with [AmDi] does not play any role in the paper and Definition 3.3 will be our working definition. \blacksquare

Theorem 3.6 *For any Borel set $A \subset X$ with $\mathbf{m}(A) < \infty$ one has $\mathcal{M}_-(A) = \text{Per}(A)$.*

Proof. Claim: $\mathcal{M}_-(A) \geq \text{Per}(A)$. We can assume with no loss of generality $\mathcal{M}_-(A) < \infty$. Let $\tau \in (0, 1)$ be fixed and let $f_r : X \rightarrow [0, 1]$ be given by

$$f_r(x) = 1 - 1 \wedge \frac{\mathbf{d}_A(x)}{r'} \quad r > 0,$$

where $r' \in (\tau r, r)$ is chosen in such a way that $\mathbf{m}(\{\mathbf{d}_A = r'\}) = 0$. Since $\text{lip}(f_r) \equiv 0$ on $\{\mathbf{d}_A > r'\}$ we obtain $\text{lip}(f_r) = 0$ \mathbf{m} -a.e. on the complement of $A^{r'}$, moreover, such slope vanishes on the interior of A as well, thus taking into account that $\partial A \subset A^r \setminus A$ for every $r > 0$ and that $\mathbf{m}(A^r \setminus A) \downarrow 0$ as $r \downarrow 0$ by our assumption $\mathcal{M}_-(A) < \infty$, we see that $\text{lip}(f_r) = 0$ \mathbf{m} -a.e. on A and thus the inequalities $\text{lip}(f_r) \leq \text{Lip}(f_r) \leq 1/r' \leq 1/(\tau r)$ give

$$\frac{\mathbf{m}(A^r) - \mathbf{m}(A)}{r} \geq \frac{\mathbf{m}(A^{r'}) - \mathbf{m}(A)}{r} \geq \tau \int_X \text{lip}(f_r) d\mathbf{m}.$$

We now estimate

$$\int_X |f_r - \chi_A| d\mathbf{m} = \int_{X \setminus A} f_r d\mathbf{m} \leq \mathbf{m}(A^r \setminus A) \rightarrow 0.$$

Then, choosing an infinitesimal sequence (r_h) of radii on which the \liminf is achieved, since $f_{r_i} \rightarrow \chi_A$ in $L^1(X, \mathbf{m})$ the very definition of $\text{Per}(A)$ gives $\mathcal{M}_-(A) \geq \tau \text{Per}(A)$. Eventually we let $\tau \uparrow 1$.

Claim: $\mathcal{M}_-^*(A) \geq \text{Per}(A)$. Let $A_h \rightarrow A$ in \mathbf{m} -measure and use the lower semicontinuity of the perimeter and the above claim to get

$$\text{Per}(A) \leq \liminf_{h \rightarrow \infty} \text{Per}(A_h) \leq \liminf_{h \rightarrow \infty} \mathcal{M}_-(A_h).$$

Since A_h are arbitrary, this proves that $\text{Per}(A) \leq \mathcal{M}_-^*(A)$.

Claim: $\text{Per}(A) \geq \mathcal{M}_-^*(A)$. Thanks to Remark 3.4 we can find a family of Lipschitz functions $f_h : X \rightarrow [0, 1]$ with $\{f_h > 0\}$ bounded, $f_h \rightarrow \chi_A$ in $L^1(X, \mathbf{m})$ and $\int_X \text{lip}(f_h) d\mathbf{m} \rightarrow \text{Per}(A)$. For all $\epsilon \in (0, 1/2)$ we can find, thanks to (3.3), $t_h \in (\epsilon, 1 - \epsilon)$ with

$$\mathcal{M}_-(\{f_h \geq t_h\}) \leq \frac{1}{1 - 2\epsilon} \int_X \text{lip}(f_h) d\mathbf{m}.$$

Since $\{f_h \geq t_h\} \rightarrow A$ in \mathbf{m} -measure it follows that $(1 - 2\epsilon)\mathcal{M}_-^*(A) \leq \text{Per}(A)$ and letting $\epsilon \downarrow 0$ the inequality is achieved. \square

4 Level sets of Lipschitz functions

In this section we study the relation between perimeter and Minkowski content for generic superlevel sets of Lipschitz functions.

Definition 3.3 of perimeter is a particular case of the following one, for $f \in L^1(X, \mathbf{m})$:

$$\text{Var}(f) := \inf \left\{ \liminf_{h \rightarrow \infty} \int_X \text{lip}(f_h) d\mathbf{m} : f_h \in \text{Lip}(X), \lim_{h \rightarrow \infty} \int_X |f_h - f| d\mathbf{m} = 0 \right\}.$$

The two concepts are closely related. Indeed, by approximating any L^1 function with step functions and, conversely, characteristic functions of $\{f > t\}$ by $\chi_\epsilon \circ f$, where χ_ϵ is a smooth approximation of $\chi_{[t, \infty)}$, it can be easily proved (see [Mi] for details) that the classical coarea formula of Fleming-Rishel, namely

$$\text{Var}(f) = \int_0^\infty \text{Per}(\{f \geq t\}) dt \quad f \in L^1(X, \mathbf{m}), f \geq 0 \quad (4.1)$$

holds even in this abstract setting, without any finiteness assumption on either side of the equality. Under the regularity assumption

$$\text{Var}(f) = \int_X \text{lip}(f) d\mathbf{m} \quad \text{for all } f \in L^1(X, \mathbf{m}) \text{ Lipschitz} \quad (4.2)$$

on the metric measure structure we can now prove that we can replace perimeter with the lower Minkowski content in (4.1), namely

$$\text{Var}(f) = \int_0^\infty \mathcal{M}_-(\{f \geq t\}) dt \quad f \in L^1(X, \mathbf{m}), f \geq 0. \quad (4.3)$$

Remark 4.1 In Euclidean spaces \mathbb{R}^n , combining the Fleming-Rishel formula, the pointwise inequalities

$$\text{Per}(\{f \geq t\}) = \mathcal{H}^{n-1}(\partial^*\{f \geq t\}) \leq \mathcal{H}^{n-1}(\partial\{f > t\}) \leq \mathcal{H}^{n-1}(\{f = t\})$$

(where $\partial^*\{f \geq t\}$ is the essential boundary of $\{f \geq t\}$, a countably \mathcal{H}^{n-1} -rectifiable set whenever $\text{Per}(\{f \geq t\})$ is finite) and the coarea formula for nonnegative Lipschitz functions

$$\int_{\mathbb{R}^n} |\nabla f| dx = \int_0^\infty \mathcal{H}^{n-1}(\{f = t\}) dt$$

one obtains

$$\text{Per}(\{f \geq t\}) = \mathcal{H}^{n-1}(\partial\{f \geq t\}) = \mathcal{H}^{n-1}(\{f = t\}) \quad \text{for } \mathcal{L}^1\text{-a.e. } t > 0$$

and that $\{f = t\}$ is countably \mathcal{H}^{n-1} -rectifiable for \mathcal{L}^1 -a.e. $t > 0$. However, the validity of (4.3) is a nontrivial information even in the Euclidean case, since rectifiability does not imply, in general, finiteness of the Minkowski content and agreement with the Hausdorff measures. See §2.13 of [AFP] for an example of compact countable set having infinite Minkowski content. Finiteness of the Minkowski contents is ensured for instance by density lower bounds of the form

$$\sigma(B_r(x)) \geq \frac{\mathbf{m}(B_r(x))}{r} \quad \forall x \in \partial A, r \in (0, 1)$$

for some finite measure σ , see Theorem 2.104 for the simple argument, which also proves that this property, in conjunction with rectifiability, provides agreement of the Minkowski contents with the Hausdorff measures. \blacksquare

Notice that, thanks to the truncation argument of Remark 3.4, an equivalent formulation of (4.2) is the lower semicontinuity of the functional $f \mapsto \int_X \text{lip}(f) d\mathbf{m}$, restricted to Lipschitz and integrable functions, w.r.t. $L^1(X, \mathbf{m})$ convergence.

Proposition 4.2 *Under assumption (4.2), one has*

$$\text{Var}(f) = \int_0^\infty \mathcal{M}_-(\{f \geq t\}) dt \tag{4.4}$$

for any Lipschitz function $f : X \rightarrow [0, \infty)$ with $\mathbf{m}(\{f > 0\}^h) < \infty$ for some $h > 0$. As a consequence

$$\mathcal{M}_-(\{f \geq t\}) = \text{Per}(\{f \geq t\}) \quad \text{for } \mathcal{L}^1\text{-a.e. } t > 0.$$

In addition, if $f : X \rightarrow [0, M]$ is 1-Lipschitz, $\mathbf{m}(\{f > 0\}^h) < \infty$ for some $h > 0$ and $\text{lip}(f) = 1$ \mathbf{m} -a.e. in $\{0 < f < M\}$, one has

$$\mathcal{M}_-(\{f \geq t\}) = \mathcal{M}_+(\{f \geq t\}) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, M).$$

Proof. Taking (4.1) and the inequality $\mathcal{M}_-(A) \geq \text{Per}(A)$ into account, for the first part of the statement it is sufficient to prove the inequality \geq in (4.4). Since by assumption $\text{Var}(f) \geq \int_X \text{lip}(f) d\mathbf{m}$, the inequality follows at once from (3.3) of Lemma 3.2.

In order to prove the second statement we start from the inequality

$$\frac{\mathbf{m}(\{f \geq t\}^h) - \mathbf{m}(\{f \geq t\})}{h} \leq \frac{\mathbf{m}(\{f \geq t - h\}) - \mathbf{m}(\{f \geq t\})}{h}$$

to get $\mathcal{M}_+(\{f \geq t\}) \leq -\mathbf{m}(\{f \geq t\})'$ for \mathcal{L}^1 -a.e. $t > 0$. By integration in $(0, M)$ we get

$$\int_0^\infty \mathcal{M}_+(\{f \geq t\}) dt \leq \mathbf{m}(\{0 < f < M\}) \leq \int_X \text{lip}(f) d\mathbf{m} = \int_0^\infty \mathcal{M}_-(\{f \geq t\}) dt,$$

whence the result follows. \square

Remark 4.3 (The case of distance functions) Assume that (X, \mathbf{d}) satisfies the following length assumption: for any $x, y \in X$ there exists an ϵ -geodesic between x and y , namely a finite collection of points z_0, \dots, z_N with $z_0 = x$, $z_N = y$, $\max_{0 \leq i < N} \mathbf{d}(z_i, z_{i+1}) < \epsilon$ and

$$\sum_{i=0}^{N-1} \mathbf{d}(z_i, z_{i+1}) < \mathbf{d}(x, y) + \epsilon.$$

If $A \subset X$ is nonempty and closed, under this length assumption on (X, \mathbf{d}) it is easy to check that $\text{lip}(\mathbf{d}_A) \equiv 1$ in $X \setminus A$. Therefore, the second part of Proposition 4.2 is applicable to all functions $(M - \mathbf{d}_A)^+$ for all $M > 0$ and provides, still under the assumption (4.2), the property

$$\mathcal{M}_-(\{\mathbf{d}_A \leq t\}) = \mathcal{M}_+(\{\mathbf{d}_A \leq t\}) = \text{Per}(\{\mathbf{d}_A \leq t\}) \quad (4.5)$$

for \mathcal{L}^1 -a.e. $t > 0$. If $\mathbf{m}(X) < \infty$ we can apply the statement also to \mathbf{d}_A to get

$$\mathcal{M}_-(\{\mathbf{d}_A \geq t\}) = \mathcal{M}_+(\{\mathbf{d}_A \geq t\}) = \text{Per}(\{\mathbf{d}_A \geq t\}) \quad (4.6)$$

for \mathcal{L}^1 -a.e. $t > 0$. Since $\text{Per}(\{\mathbf{d}_A \geq t\}) = \text{Per}(\{\mathbf{d}_A \leq t\})$ with at most countably many exceptions, the quantities in (4.5) and (4.6) coincide \mathcal{L}^1 -a.e. in $(0, \infty)$.

If the length assumption above is enforced to

$$\forall x, y \in X, \forall \epsilon > 0 \text{ there exists } \gamma \in C([0, 1]; X) \text{ with } \gamma_0 = x, \gamma_1 = y, \text{length}(\gamma) < \mathbf{d}(x, y) + \epsilon$$

(notice that the two length properties are equivalent in separable spaces), a simple continuity argument gives

$$\{\mathbf{d}_A = t\}^r \setminus \{\mathbf{d}_A = t\} = \{0 < \mathbf{d}_{\{\mathbf{d}_A \geq t\}} < r\} \cup \{0 < \mathbf{d}_{\{\mathbf{d}_A \leq t\}} < r\} \quad \forall t \in (0, \infty), r > 0.$$

Then, since $\mathbf{m}(\{\mathbf{d}_A = t\}) = 0$ with at most countably many exceptions, we obtain existence of the Minkowski content of the level set $\{\mathbf{d}_A = t\}$

$$\lim_{r \downarrow 0} \frac{\mathbf{m}(\{\mathbf{d}_A = t\}^r)}{2r} = \text{Per}(\{\mathbf{d}_A \geq t\}) = \text{Per}(\{\mathbf{d}_A \leq t\})$$

for \mathcal{L}^1 -a.e. $t > 0$. \blacksquare

Concerning assumption (4.2), we point that it holds on doubling $\text{RCD}(K, \infty)$ spaces (see [AGS11b]) for the definition of the latter. Indeed, on one side we know by Rajala's paper [Ra] that on these spaces a 1-1 weak Poincaré inequality holds, so that thanks to Cheeger's results in [Chee] we have that the p -weak gradient is \mathbf{m} -a.e. equal to the local Lipschitz constant of Lipschitz functions for $p > 1$. On the other hand, in [GiHa] it has been proved that on $\text{RCD}(K, \infty)$ spaces the p -weak gradients all coincide even for $p = 1$ (this result is based on the Bakry-Emery estimate in the form established by Savaré in [Sa]), thus giving the claim. We illustrate in the next remark the known relations between $\text{lip}(f)$ and other weak notions of gradient, and why doubling&1-Poincaré alone are not sufficient to ensure the validity of (4.2).

Remark 4.4 Recall that one of the key results of the seminal paper [Chee] is the lower semicontinuity of the functional $\int_X \text{lip}(f)^p d\mathbf{m}$ w.r.t. L^p convergence on locally Lipschitz functions, for all $p \in (1, \infty)$, assuming the doubling property of the metric measure structure and the validity of a p -Poincaré inequality. This result can also be rephrased by saying that $\text{lip}(f)$ coincides \mathbf{m} -a.e. with the minimal p -weak upper gradient.

However, under the same structural assumptions, the result is not true in general when $p = 1$. Indeed, revisiting an example by Carbone-Sbordone [CaSb], in [HKLL14], a metric measure space (X, d, \mathbf{m}) is built as follows: $X = [0, 1]$ is endowed with the Euclidean distance and with the weighted Lebesgue measure $\mathbf{m} = \omega \mathcal{L}^1$, with $\omega \equiv 1$ on a “fat” Cantor set $K \subset (0, 1)$ (i.e. a compact totally disconnected set with positive Lebesgue measure) and $\omega = 1/2$, say, on $(0, 1) \setminus K$. It is clear that (X, d, \mathbf{m}) , being comparable to the standard Euclidean structure, satisfies the doubling and 1-Poincaré assumptions. On the other hand, in [HKLL14] Lipschitz functions f_n convergent to the identity function f in $L^1(X, \mathbf{m})$ are built in such a way that

$$\limsup_{n \rightarrow \infty} \int_X \text{lip}(f_n) d\mathbf{m} < \int_X \text{lip}(f) d\mathbf{m},$$

thus disproving the lower semicontinuity. This corresponds to the fact that two notions of minimal 1-weak upper gradients are possible in the $H^{1,1}$ (and BV) theory. The two notions do not coincide in general, and $\text{lip}(f)$ coincides only with one of them (see [HKST, Theorem 12.5.1]). See also [AmPiSp] and [HKLL14] for a more detailed discussion. ■

5 Some open problems

In complete metric measure spaces with a doubling measure, under the assumption of the validity of a 1-Poincaré inequality, the first author proved in [Am1], [AM2] that the perimeter measure of Remark 3.5 coincides with $\theta \mathcal{S}^h \llcorner \partial^* E$, where $\partial^* E$ is the essential boundary of E (i.e. the complement of the union of density and rarefaction points of E) and \mathcal{S}^h is the measure built out of the gauge function $\zeta(\overline{B}_r(x)) = \mathbf{m}(\overline{B}_r(x))/(2r)$ with Carathéodory’s construction. The density θ is bounded from below by the structural constants involved in the doubling and Poincaré assumptions, but little more is known in general about it. Under additional regularity assumptions, it would be interesting to relate more closely the Minkowski content of E with the measure $\mathcal{S}^h \llcorner \partial^* E$, as in the Euclidean-Riemannian theory.

In the same vein, one can prove the coarea inequality (Proposition 5.1 in [AM2])

$$\int_0^\infty \mathcal{S}^h(B \cap \{f = t\}) dt \leq \text{Lip}(f) \mathbf{m}(B) \quad B \text{ Borel} \quad (5.1)$$

reminiscent of (4.4): we propose here a self-included improved and localized version of this inequality

Proposition 5.1 *Let $f : X \rightarrow [0, \infty)$ be a Lipschitz function and let \mathbf{m} be a doubling measure on X such that $\mathbf{m}\{f > 0\} < \infty$. Then we have that for every B borel*

$$\begin{aligned} \int_0^\infty \mathcal{S}^h(B \cap \{f = t\}) dt &\leq 2 \int_B \text{lip}(f) d\mathbf{m}, \\ \int_0^\infty \mathcal{S}^h(B \cap \{f = t\}) dt &\leq \int_B \text{lip}_a(f) d\mathbf{m}. \end{aligned}$$

Proof. First we recall that if \mathbf{m} is a doubling measure then these two properties hold, which are both consequences of the Vitali covering theorem and the doubling property:

- (i) $\mathbf{m}(B) = \inf \{ \sum_i \mathbf{m}(B_i) : B_i = B(x_i, r_i), r_i \leq \delta, B \subseteq \bigcup B_i, \}$;
- (ii) For every collection of balls \mathcal{F} that is a fine covering of B we have a disjoint covering \mathcal{F}' that covers \mathbf{m} -almost every B .

For every δ and every ε we have, for property (i), the existence of balls B_i such that $B \subseteq \bigcup B_i$ and $\mathbf{m}(B) \leq \varepsilon + \sum \mathbf{m}(B_i)$. Let us denote $t_i^- = \inf_{B_i} f(x)$ and $t_i^+ = \sup_{B_i} f(x)$: then we have that $B_i \cap \{f = t\} \neq \emptyset$ implies for sure that $t \in [t_i^-, t_i^+]$ and in particular we have

$$\mathcal{S}_\delta^h(B \cap \{f = t\}) \leq \sum_{\substack{i \text{ s.t.} \\ t \in [t_i^-, t_i^+]}} \frac{\mathbf{m}(\overline{B_i})}{2r_i}.$$

Integrating this inequality from 0 to ∞ we have that

$$\int_0^\infty \mathcal{S}_\delta^h(B \cap \{f = t\}) dt \leq \sum_i \frac{\mathbf{m}(\overline{B_i})}{2r_i} (t_i^+ - t_i^-). \quad (5.2)$$

It is clear that $(t_i^+ - t_i^-) \leq \text{Lip}(f)2r_i$ and thus we obtain

$$\int_0^\infty \mathcal{S}_\delta^h(B \cap \{f = t\}) dt \leq \text{Lip}(f) \sum_i \mathbf{m}(B_i) \leq \text{Lip}(f)(\mathbf{m}(B) + \varepsilon).$$

Letting $\varepsilon, \delta \rightarrow 0$ we will get (5.1). This let us prove that if $\mathbf{m}(B) = 0$ then $\mathcal{S}_\delta^h(B \cap \{f = t\}) = 0$ for almost every t ; a fortiori, for every δ , we have $\mathcal{S}_\delta^h(B \cap \{f = t\}) = 0$ for almost every t .

Now we use property (ii): for every δ we find a disjoint family of balls $\{B_i\}_{i \in \mathbb{N}}$ with radii less than δ such that $\mathbf{m}(B \setminus \bigcup_i B_i) = 0$; let us call $\tilde{B} = \bigcup_i B_i$. Since $\mathbf{m}(B \setminus \tilde{B}) = 0$ we have that

$$\mathcal{S}_\delta^h(B \cap \{f = t\}) \leq \mathcal{S}_\delta^h(\tilde{B} \cap \{f = t\}) + \mathcal{S}_\delta^h((B \setminus \tilde{B}) \cap \{f = t\}) = \mathcal{S}_\delta^h(\tilde{B} \cap \{f = t\}) \quad \text{for a.e. } t,$$

and thus $\int_0^\infty \mathcal{S}_\delta^h(B \cap \{f = t\}) dt = \int_0^\infty \mathcal{S}_\delta^h(\tilde{B} \cap \{f = t\}) dt$. Now we can apply (5.2) for \tilde{B} and its covering to obtain the same inequality for B . Now we notice now that if $x \in B_i$ we have that

$$\begin{aligned} \frac{t_i^+ - t_i^-}{2r_i} &= \sup_{y, y' \in B_i} \frac{f(y) - f(y')}{2r_i} \leq \sup_{y, y' \in B_i} \frac{|f(y) - f(x)| + |f(y') - f(x)|}{2r_i} \\ &\leq 2 \sup_{d(x, y) \leq 2\delta} \frac{|f(y) - f(x)|}{d(y, x)} =: 2\text{sl}(x, 2\delta) \end{aligned}$$

where sl is the local slope that satisfies $\lim_{\delta \rightarrow 0} \text{sl}(x, \delta) = \text{lip}(f)(x)$. In particular, using that B_i are disjoint and cover \mathbf{m} -almost B , we have deduce from (5.2) and the last inequality that

$$\int_0^\infty \mathcal{S}_\delta^h(B \cap \{f = t\}) dt \leq 2 \int_B \text{sl}(x, 2\delta) d\mathbf{m}. \quad (5.3)$$

Now, we conclude letting $\delta \rightarrow 0$. We can obtain the result for lip_a using that, for $x \in B_i$,

$$\frac{t_i^+ - t_i^-}{2r_i} \leq \text{Lip}(f, B_i) \leq \text{Lip}(f, B(x, 2\delta)) \rightarrow \text{lip}_a(f).$$

□

Corollary 5.2 *Let (X, d, m) be a doubling metric measure space such that (4.2) holds. Then for every Lipschitz function $f : X \rightarrow [0, \infty)$ such that $m\{f > 0\} < \infty$ we have that for almost every $t > 0$ the set $E_t = \{f \geq t\}$ is of finite perimeter and $P(E_t, \cdot) \geq \frac{1}{2} \mathcal{S}^h \llcorner \partial E_t$.*

It would be nice to improve it, replacing $\text{Lip}(f)m(B)$ with $\int_B \text{lip}(f) dm$ in the right hand side of (5.1), but it seems difficult to obtain. We notice also that in this inequality is not optimal since, in the euclidean case $X = \mathbb{R}^d$, we have $\mathcal{S}^h = \frac{\omega_d}{2\omega_{d-1}} \mathcal{H}^{d-1}$.

Finally, a natural question is the identification of regularity assumptions ensuring equality between Minkowski content and perimeter, somehow bypassing the question of connecting these notions to $\mathcal{S}^h \llcorner \partial^* E$.

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